ISI – Bangalore Center – B Math - Physics I – Mid Semester Exam
Date: 22 February 2017. Duration of Exam: 3 hours
Total marks: 70

Q1. [Total Marks: 2+2+2+3+3+2=14]

Please give BRIEF answers to the following.

1a.) State the necessary and sufficient condition for a force $\vec{F}(x, y, z)$ to be conservative, i.e. the work done by the force on a particle that moves from an initial position to a final position is independent of the path taken.

1b.) Determine which of the following force fields is conservative.
$\vec{F}_1 = -2x\vec{i} - 2y\vec{j} - 2z\vec{k}$, $\vec{F}_2 = y\vec{i} - x\vec{j}$

1c.) A particle is moving under a force $\vec{F}(r,\theta,\varphi) = (e^{-r}\cos\theta\sin\varphi)\hat{r}$. Determine if the angular momentum is conserved or not.

1d.) Suppose that the spring constant k of a spring changes with time. How is the rate of change of mechanical energy related to k? (Provide a mathematical relation. Detailed derivation not necessary)

1e.) A very slightly damped harmonic oscillator with natural frequency ω is being driven by an external force $F(t) = f\sin\omega t\cos\omega t$. Which of the following options is the correct behavior of the position of the oscillator as a function of time? Give a brief justification of the answer.

- It will approach $x=0$
- It will oscillate with frequency ω
- It will oscillate with frequency 2ω
- It will oscillate with frequency 3ω

1f.) A proton is moving from far away towards a stationary proton. The interaction between them is according to the standard Coulomb's law. Under what conditions can the moving proton be captured by the fixed proton into an orbit around it? Briefly explain your answer.

Q2. [Total Marks: 1+3+3+3+1+3=14]

Assume that one end of a spring with spring constant k is fixed at the origin and a particle of mass *m* is attached at the other end of the spring. The particle is free to move in all there dimensions. The unstretched length of the spring is *l*. Ignore gravity for this problem.

2a.) Write an expression for the force on the particle if it is stretched to a position $\vec{r}(t)$.
2b.) Prove that the particle attached to the spring will move in a plane when released from any initial position with any initial velocity.
2c.) Using polar coordinates r,θ show that the total energy of the system can be written as $E=\frac{1}{2}m\dot{r}^2+V_{eff}(r)$. Your expression for $V_{eff}(r)$ should contain only m,k,l,r and L where L is the magnitude of the angular momentum $\vec{L}$).

2d.) Plot $V_{eff}(r)$ and show that all trajectories are bounded. Show that $V_{eff}(r)$ has one stable minimum which depends on the value of L (and of course also on k,l etc.).

2e.) What kind of orbit will the mass attached to the spring have if its energy is equal to V_{eff}? (You need NOT derive a formula for the orbit to answer this.)

2f.) Suppose the minimum of $V_{eff}(r)$ is located at $r=\frac{5}{4}l$, and its energy is equal to $V_{eff}(r=\frac{5}{4}l)$ then determine $r(t),\theta(t)$ for all t.

Q3. [Total Marks: 6+8=14]

3a.) A un-damped one dimensional simple harmonic oscillator of mass *m* and spring constant k is subjected to an external force F given by
$F=0$ for $t\le 0$, $F=f$ for $t>0$ where $f>0$ is a constant.

Show that if the particle is initially at rest at the origin then at subsequent times, its position is given by $x(t>0)=\frac{f}{k}[1-\cos\omega t]$ where $\omega^2=k/m$

3b.)Suppose a particle of mass *m* is hung with a spring with spring constant k. Let x_0 be the new equilibrium position (where *x* is measured down). Suppose a constant downward force is applied from $t=0$ to $t=t_0, t_0>0$. Show that the displacement of the mass at time $t>t_0$ is given by

$$x=x_0+\frac{F}{k}[\cos\omega(t-t_0)-\cos\omega t]\text{ where }\omega^2=k/m.$$

[HINT: it will help to use variable $y=x-x_0$, and you can use the result in part a.).]

Q4. [Total Marks: 5+5+4=14]

4a.) Derive the path equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{L^2u^2}$$ where L is the angular momentum,

for a particle of unit mass moving in a central force given by $\vec{F} = f(r)\hat{r}$.

4b.) If $f = -\frac{\gamma}{r^3}$, find the solutions for the orbits for three distinct cases,

$L^2 < \gamma, L^2 = \gamma, L^2 = \gamma$

4c.) If $L^2 = \frac{9}{8}\gamma$, and if the initial condition is $r = \infty, \theta = 0$, then show that the solution is $u = B\left(\sin\frac{\theta}{3}\right)$ where B is a constant. Plot the trajectory of the moving particle. For this purpose you can assume that if there was no force, then the particle moving in a straight line would have missed the origin by the perpendicular distance p.

Q5. [Total Marks: 5+4+5=14]

A particle is moving in one dimension under the potential

$V(x) = \frac{1}{2}kx^2 + \frac{a}{2x^2}$ where k and a are positive constants. Assume that the particle is initially located in the x>0 region.

5a.) Determine the location of the minima of the potential.

Using an approximation to the potential in terms of its Taylor series expansion around its minimum and keeping upto the second derivative terms, show that the equation of motion around the minimum location is that of a simple harmonic oscillator.

Determine the period of oscillation in this approximation.

5b.) Calculate the turning points of the particle if it has total energy E.

5c.) Calculate the period of oscillation with an arbitrary value of E and show that it is independent of E. How is the period of oscillation compared to the period found in a.)?

[HINT: You may need to use the indefinite integral

$\int dx \frac{1}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \ln[2ax + b + 2\sqrt{a(ax^2 + bx + c)}]$. Also notice that that the limits of the integral in this problem are solutions to the zeroes of the expression $ax^2 + bx + c$ in the integrand.]